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Multicomponent nonlinear dynamical systems inspired by the Toda lattice

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Abstract
A matrix nonlinear model on an infinite lattice is proposed. The model admits the Lax representation as being the natural matrix generalization of the ordinary Toda lattice. As a result the system is exactly integrable and possesses the Bäcklund auto-transformation useful in generating nontrivial solutions. The auxiliary matrix integrable nonlinear system reducible to the Bäcklund auto-transformation of interest is developed. The simplest regular and nonregular solutions of the basic model are found and analysed. Thus, in parallel with the kink-type field component the regular solution was observed to consist of field components evolving as solitons on finite backgrounds. In a particular case of four-component realization, the model permits both Lagrangian and Hamiltonian formulations, while the corresponding Bäcklund transformation is proved to be the canonical one. We believe that similar statements can be made for any multi-component version of the general matrix model, provided an appropriate parametrization of field variables is adopted.

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1. Introduction

It is well known [1] that the Toda system [2–5]

$$\ddot{q}(n) = \exp[q(n+1) - q(n)] - \exp[q(n) - q(n-1)] \tag{1}$$

can be obtained from the Lax equation

$$\dot{L}(n|\lambda) = A(n+1|\lambda)L(n|\lambda) - L(n|\lambda)A(n|\lambda) \tag{2}$$

under the assumption that the spectral operator $L(n|\lambda)$ is taken to be

$$L(n|\lambda) = \begin{pmatrix} p(n) + \lambda & \mathrm{i}\exp[+q(n)] \\ \mathrm{i}\exp[-q(n)] & 0 \end{pmatrix} \tag{3}$$

while the evolution operator $A(n|\lambda)$ is sought in the form

$$A(n|\lambda) = \begin{pmatrix} 0 & A_{12}(n) \\ A_{21}(n) & \lambda \end{pmatrix}. \tag{4}$$

This model is characterized by a single field component $q(n)$ implying the longitudinal displacement of the nth molecule in an infinite chain ($-\infty < n < +\infty$) with exponential inter-molecular interaction. The overdot stands for the derivative with respect to time τ whereas the spectral parameter λ is understood to be time-independent.

The question that arises is what systems could be thought of as the most natural generalizations of the one-component Toda lattice (1) to the multi-component case, or more precisely what guiding principle should be chosen to derive them?

2. General matrix model and its four-component realization

Observing that the product of the off-diagonal elements of the Toda spectral operator (3) is equal to -1 we impose the similar restriction

$$F(n)G(n) = I \tag{5}$$

on the off-diagonal $M \times M$ submatrices $\mathrm{i}F(n)$ and $\mathrm{i}G(n)$ of its $2M \times 2M$ matrix extension

$$L(n|\lambda) = \begin{pmatrix} \Pi(n) + \lambda \cdot I & \mathrm{i}F(n) \\ \mathrm{i}G(n) & 0 \cdot I \end{pmatrix}. \tag{6}$$

Here $\Pi(n)$ is the momentum $M \times M$ submatrix and I is the unity $M \times M$ submatrix. The off-diagonal $M \times M$ submatrices $\overset{-+}{A}(n)$ and $\overset{+-}{A}(n)$ of the extended evolution operator

$$A(n|\lambda) = \begin{pmatrix} 0 \cdot I & \overset{-+}{A}(n) \\ \overset{+-}{A}(n) & \lambda \cdot I \end{pmatrix} \tag{7}$$

have to be determined from the Lax equation (2) and are equal to

$$\overset{-+}{A}(n) = -\mathrm{i}F(n) \tag{8}$$

$$\overset{+-}{A}(n) = -\mathrm{i}G(n-1). \tag{9}$$

The evolution equations

$$\dot{\Pi}(n) = F(n+1)G(n) - F(n)G(n-1) \tag{10}$$

$$\tfrac{1}{2}[\dot{F}(n)G(n) - F(n)\dot{G}(n)] = \Pi(n) \tag{11}$$

for the momentum submatrix $\Pi(n)$ and mutually inverse coordinate submatrices $F(n)$ and $G(n)$ can be readily isolated from the Lax equation (2) too. Unification of these two equations, (10) and (11), gives rise to the general $M \times M$ matrix model

$$\tfrac{1}{2}[\ddot{F}(n)G(n) - F(n)\ddot{G}(n)] = F(n+1)G(n) - F(n)G(n-1) \tag{12}$$

which at $M = 1$ and parametrization $F(n) = \exp[+q(n)]$, $G(n) = \exp[-q(n)]$ is seen to be reduced to the standard Toda system (1).

At $M = 2$ the submatrices $F(n)$ and $G(n)$ can be parametrized as

$$F_{11}(n) = \mathrm{ch}\,\rho(n)\exp[+w(n) + \alpha(n)] \tag{13}$$

$$F_{12}(n) = \mathrm{sh}\,\rho(n)\exp[+w(n) + \beta(n)] \tag{14}$$

$$F_{21}(n) = \mathrm{sh}\,\rho(n)\exp[+w(n) - \beta(n)] \tag{15}$$

$$F_{22}(n) = \mathrm{ch}\,\rho(n)\exp[+w(n) - \alpha(n)] \tag{16}$$

$$G_{11}(n) = \operatorname{ch}\rho(n)\exp[-w(n) - \alpha(n)] \tag{17}$$
$$G_{12}(n) = -\operatorname{sh}\rho(n)\exp[-w(n) + \beta(n)] \tag{18}$$
$$G_{21}(n) = -\operatorname{sh}\rho(n)\exp[-w(n) - \beta(n)] \tag{19}$$
$$G_{22}(n) = \operatorname{ch}\rho(n)\exp[-w(n) + \alpha(n)] \tag{20}$$

yielding the model (12) to be presented in explicit Lagrangian form

$$\frac{\mathrm{d}}{\mathrm{d}\tau}[\partial\mathcal{L}/\partial\dot{w}(n)] = \partial\mathcal{L}/\partial w(n) \tag{21}$$
$$\frac{\mathrm{d}}{\mathrm{d}\tau}[\partial\mathcal{L}/\partial\dot{\rho}(n)] = \partial\mathcal{L}/\partial\rho(n) \tag{22}$$
$$\frac{\mathrm{d}}{\mathrm{d}\tau}[\partial\mathcal{L}/\partial\dot{\alpha}(n)] = \partial\mathcal{L}/\partial\alpha(n) \tag{23}$$
$$\frac{\mathrm{d}}{\mathrm{d}\tau}[\partial\mathcal{L}/\partial\dot{\beta}(n)] = \partial\mathcal{L}/\partial\beta(n) \tag{24}$$

with the Lagrangian function $\mathcal{L}$ given by

$$\begin{aligned}\mathcal{L} = \frac{1}{2}\sum_{m=-\infty}^{\infty}&[\dot{w}^2(m) + \dot{\rho}^2(m) + \dot{\alpha}^2(m)\operatorname{ch}^2\rho(m) - \dot{\beta}^2(m)\operatorname{sh}^2\rho(m)]\\ &- \sum_{m=-\infty}^{\infty}\operatorname{ch}\rho(m)\operatorname{ch}\rho(m-1)\operatorname{ch}[\alpha(m) - \alpha(m-1)]\exp[w(m) - w(m-1)]\\ &+ \sum_{m=-\infty}^{\infty}\operatorname{sh}\rho(m)\operatorname{sh}\rho(m-1)\operatorname{ch}[\beta(m) - \beta(m-1)]\exp[w(m) - w(m-1)].\end{aligned} \tag{25}$$

Introducing the field momenta

$$p_w(n) = \partial\mathcal{L}/\partial\dot{w}(n) = \dot{w}(n) \tag{26}$$
$$p_\rho(n) = \partial\mathcal{L}/\partial\dot{\rho}(n) = \dot{\rho}(n) \tag{27}$$
$$p_\alpha(n) = \partial\mathcal{L}/\partial\dot{\alpha}(n) = \dot{\alpha}(n)\operatorname{ch}^2\rho(n) \tag{28}$$
$$p_\beta(n) = \partial\mathcal{L}/\partial\dot{\beta}(n) = -\dot{\beta}(n)\operatorname{sh}^2\rho(n) \tag{29}$$

conjugated canonically to the field coordinates $w(n)$, $\rho(n)$, $\alpha(n)$, $\beta(n)$ the system (21)–(25) can be easily converted into the standard Hamiltonian form

$$\dot{w}(n) = \partial\mathcal{H}/\partial p_w(n) \qquad \dot{p}_w(n) = -\partial\mathcal{H}/\partial w(n) \tag{30}$$
$$\dot{\rho}(n) = \partial\mathcal{H}/\partial p_\rho(n) \qquad \dot{p}_\rho(n) = -\partial\mathcal{H}/\partial\rho(n) \tag{31}$$
$$\dot{\alpha}(n) = \partial\mathcal{H}/\partial p_\alpha(n) \qquad \dot{p}_\alpha(n) = -\partial\mathcal{H}/\partial\alpha(n) \tag{32}$$
$$\dot{\beta}(n) = \partial\mathcal{H}/\partial p_\beta(n) \qquad \dot{p}_\beta(n) = -\partial\mathcal{H}/\partial\beta(n) \tag{33}$$

with the Hamiltonian function $\mathcal{H}$ defined by the expression

$$\begin{aligned}\mathcal{H} = \frac{1}{2}\sum_{m=-\infty}^{\infty}&\left[p_w^2(m) + p_\rho^2(m) + p_\alpha^2(m)\big/\operatorname{ch}^2\rho(m) - p_\beta^2(m)\big/\operatorname{sh}^2\rho(m)\right]\\ &+ \sum_{m=-\infty}^{\infty}\operatorname{ch}\rho(m)\operatorname{ch}\rho(m-1)\operatorname{ch}[\alpha(m) - \alpha(m-1)]\exp[w(m) - w(m-1)]\\ &- \sum_{m=-\infty}^{\infty}\operatorname{sh}\rho(m)\operatorname{sh}\rho(m-1)\operatorname{ch}[\beta(m) - \beta(m-1)]\exp[w(m) - w(m-1)].\end{aligned} \tag{34}$$

Thus, at least in its four-component version ($M = 2$), the model (12) is proved to be a typical dynamical system admitting both Lagrangian (21)–(25) and Hamiltonian (30)–(34) embodiments. In what follows, we will use the suggested four-component representation for illustrative purposes.

Meanwhile, what really matters is the mere fact of an exact integrability of the general model (12) arising from its explicit representation in the Lax form (2). Here we will endeavour to generate the simplest solutions via the Bäcklund auto-transformation regarding the model under study (12). In doing so, we begin by developing an auxiliary integrable nonlinear model leading in a straightforward manner to the Bäcklund auto-transformation of interest. Such a course of actions has been prompted by the close relationship between the Toda system (1) and the auxiliary Kac–van Moerbeke model [6] on the same Bäcklund-transforming subject [7].

3. Auxiliary integrable nonlinear model

Let us suppose the auxiliary nonlinear problem to be integrable, i.e. it can be written in the Lax form (2).

The next step is the most crucial one and comprises a fortunate choice of spectral operator $L(n|\lambda)$. After several unsuccessful attempts we have finally found it to be as follows:

$$L(n|\lambda) = \begin{pmatrix} \lambda^2 \cdot I - F_+(n)G_-(n) & \mathrm{i}\lambda F_-(n) \\ \mathrm{i}\lambda G_+(n) & -G_+(n)F_-(n) \end{pmatrix}. \tag{35}$$

Here $F_-(n)$, $G_-(n)$ and $F_+(n)$, $G_+(n)$ are the $M \times M$ submatrices linked by the constraints

$$F_-(n)G_-(n) = I \tag{36}$$

$$F_+(n)G_+(n) = I \tag{37}$$

while I denotes the $M \times M$ unity submatrix and λ stands for the time-independent spectral parameter as previously.

Now all subsequent calculations become practically predetermined.

Indeed, seeking the evolution operator $A(n|\lambda)$ in the form

$$A(n|\lambda) = \begin{pmatrix} \overset{--}{\Gamma}(n) & \lambda \overset{-+}{\Gamma}(n) \\ \lambda \overset{+-}{\Gamma}(n) & \lambda^2 \cdot I + \overset{++}{\Gamma}(n) \end{pmatrix} \tag{38}$$

and using the Lax equation (2) we are able both to restore the involved $M \times M$ submatrices

$$\overset{--}{\Gamma}(n) = -F_-(n)G_+(n-1) \tag{39}$$

$$\overset{-+}{\Gamma}(n) = -\mathrm{i}F_-(n) \tag{40}$$

$$\overset{+-}{\Gamma}(n) = -\mathrm{i}G_+(n-1) \tag{41}$$

$$\overset{++}{A}(n) = \Gamma + G_+(n-1)F_-(n) \tag{42}$$

and to isolate the integrable nonlinear model of interest

$$\dot{F}_-(n) = -F_-(n)\Gamma - F_-(n)G_+(n-1)F_-(n) - F_+(n) \tag{43}$$

$$\dot{G}_-(n) = +\Gamma G_-(n) + G_-(n)F_+(n)G_-(n) + G_+(n-1) \tag{44}$$

$$\dot{F}_+(n) = -F_+(n)\Gamma - F_+(n)G_-(n)F_+(n) - F_-(n+1) \tag{45}$$

$$\dot{G}_+(n) = +\Gamma G_+(n) + G_+(n)F_-(n+1)G_+(n) + G_-(n). \tag{46}$$

Here the $M \times M$ submatrix Γ is not bound to be time-independent although its coordinate independence must be taken for granted.

In view of the invertibility constraints (36) and (37), equations (43)–(46) are not independent and can be painlessly compressed into just two formulae:

$$\frac{1}{2}[\dot{F}_-(n)G_-(n) - F_-(n)\dot{G}_-(n)] \\ = -F_-(n)\Gamma G_-(n) - F_+(n)G_-(n) - F_-(n)G_+(n-1) \tag{47}$$

$$\frac{1}{2}[\dot{F}_+(n)G_+(n) - F_+(n)\dot{G}_+(n)] \\ = -F_+(n)\Gamma G_+(n) - F_+(n)G_-(n) - F_-(n+1)G_+(n). \tag{48}$$

This final form of the auxiliary model will be seen to manifest itself as a direct predecessor of the Bäcklund auto-transformation announced at the end of the previous section.

4. Bäcklund auto-transformation

Differentiating the final form of the auxiliary nonlinear model (47) and (48) with respect to time τ and removing the first derivatives $\dot{F}_-(n)$, $\dot{G}_-(n)$ and $\dot{F}_+(n)$, $\dot{G}_+(n)$ by means of its original form (43)–(46) it can be easily observed that the evolution of either of the two pairs of matrices $F_-(n)$, $G_-(n)$ and $F_+(n)$, $G_+(n)$ is governed by the same self-consistent matrix equation provided that the submatrix Γ is a multiple of the unity submatrix

$$\Gamma = -\gamma \cdot I \tag{49}$$

with γ supposed to be an arbitrary constant c-number. Each such equation coincides literally with the general matrix model of our main interest (12).

As a consequence, plugging $\Gamma = -\gamma \cdot I$ into the auxiliary model (47) and (48) we inevitably come to the matrix Bäcklund transformation

$$\frac{1}{2}[\dot{F}_-(n)G_-(n) - F_-(n)\dot{G}_-(n)] = \gamma \cdot I - F_+(n)G_-(n) - F_-(n)G_+(n-1) \tag{50}$$

$$\frac{1}{2}[\dot{F}_+(n)G_+(n) - F_+(n)\dot{G}_+(n)] = \gamma \cdot I - F_+(n)G_-(n) - F_-(n+1)G_+(n) \tag{51}$$

tying together the two different solutions $F_-(n)$, G_-(n) and $F_+(n)$, $G_+(n)$ of the basic model (12).

At $M = 1$ when the parametrization $F_\mp(n) = \exp[+q_\mp(n)]$ and $G_\mp(n) = \exp[-q_\mp(n)]$ is assumed, the obtained formulae (50) and (51) are seen to realize the Bäcklund auto-transformation for the Toda lattice suggested by Wadati and Toda [8, 9].

At $M = 2$ the situation becomes somewhat complicated but is still traced explicitly. Specifically adopting for $F_-(n)$, $G_-(n)$ and $F_+(n)$, $G_+(n)$ the parametrization similar to that given by the expressions (13)–(20) (with the sublabels − and + to be added respectively) we have

$$\dot{w}_-(n) = p_{w_-}(n) = \partial\mathcal{G}/\partial w_-(n) \tag{52}$$

$$\dot{\rho}_-(n) = p_{\rho_-}(n) = \partial\mathcal{G}/\partial\rho_-(n) \tag{53}$$

$$\dot{\alpha}_-(n)\,\mathrm{ch}^2\rho_-(n) = p_{\alpha_-}(n) = \partial\mathcal{G}/\partial\alpha_-(n) \tag{54}$$

$$-\dot{\beta}_-(n)\,\mathrm{sh}^2\rho_-(n) = p_{\beta_-}(n) = \partial\mathcal{G}/\partial\beta_-(n) \tag{55}$$

$$\dot{w}_+(n) = p_{w_+}(n) = -\partial\mathcal{G}/\partial w_+(n) \tag{56}$$

$$\dot{\rho}_+(n) = p_{\rho_+}(n) = -\partial\mathcal{G}/\partial\rho_+(n) \tag{57}$$

$$\dot{\alpha}_+(n)\,\mathrm{ch}^2\rho_+(n) = p_{\alpha_+}(n) = -\partial\mathcal{G}/\partial\alpha_+(n) \tag{58}$$

$$-\dot{\beta}_+(n)\,\mathrm{sh}^2\rho_+(n) = p_{\beta_+}(n) = -\partial\mathcal{G}/\partial\beta_+(n). \tag{59}$$

Here the quantity

$$
\begin{aligned}
\mathcal{G} = \sum_{m=-\infty}^{\infty} \gamma[w_-(m) - w_+(m)] \\
&+ \sum_{m=-\infty}^{\infty} \operatorname{ch}\rho_-(m)\operatorname{ch}\rho_+(m)\operatorname{ch}[\alpha_-(m) - \alpha_+(m)]\exp[-w_-(m) + w_+(m)] \\
&- \sum_{m=-\infty}^{\infty} \operatorname{sh}\rho_-(m)\operatorname{sh}\rho_+(m)\operatorname{ch}[\beta_-(m) - \beta_+(m)]\exp[-w_-(m) + w_+(m)] \\
&- \sum_{m=-\infty}^{\infty} \operatorname{ch}\rho_-(m)\operatorname{ch}\rho_+(m-1)\operatorname{ch}[\alpha_-(m) - \alpha_+(m-1)] \\
&\times \exp[w_-(m) - w_+(m-1)] \\
&+ \sum_{m=-\infty}^{\infty} \operatorname{sh}\rho_-(m)\operatorname{sh}\rho_+(m-1)\operatorname{ch}[\beta_-(m) - \beta_+(m-1)] \\
&\times \exp[w_-(m) - w_+(m-1)]
\end{aligned} \tag{60}
$$

turns out to be nothing but the generating function establishing the canonical transformation between minus (−) and plus (+) labelled dynamical variables of the same four-component Hamiltonian nonlinear system (30)–(34).

5. Simplest nontrivial solutions

Let us generate the simplest nontrivial solutions of the general matrix nonlinear model (12) starting with the trivial one

$$F_-(n) = F \qquad G_-(n) = G. \tag{61}$$

Here F and G are the time- and coordinate-independent $M \times M$ matrices linked by the natural constriction

$$FG = I \tag{62}$$

but are arbitrary in all other respects.

In doing so, we insert the trivial solution (61) into the general Bäcklund transforming equations (50) and (51) and rearranging the second equation via the first one, we obtain

$$F(n)G + FG(n-1) = \gamma \cdot I \tag{63}$$

$$\tfrac{1}{2}[\dot{F}(n)G(n) - F(n)\dot{G}(n)] = FG(n-1) - FG(n). \tag{64}$$

Here $F_+(n)$ and $G_+(n)$ have been renamed $F(n)$ and $G(n)$, respectively, for the sake of brevity.

To proceed further, it is reasonable to rely upon the ansatz

$$F(n) = \Phi^{-1}(n)\Phi(n+1)F \tag{65}$$

$$G(n) = G\Phi^{-1}(n+1)\Phi(n) \tag{66}$$

which having been plugged into equations (63) and (64) yields

$$\Phi(n+1) + \Phi(n-1) = \gamma\Phi(n) \tag{67}$$

$$\dot{\Phi}(n+1)\Phi^{-1}(n+1) + \Phi(n)\Phi^{-1}(n+1) = \dot{\Phi}(n)\Phi^{-1}(n) + \Phi(n-1)\Phi^{-1}(n). \tag{68}$$

Here $\Phi(n)$ stands for the unknown $M \times M$ matrix function of coordinate n and time τ.

The second of these equations (68) is found to be equivalent to

$$\dot{\Phi}(n) = C\Phi(n) - \Phi(n-1) \tag{69}$$

where C denotes an arbitrary coordinate-independent $M \times M$ matrix with an arbitrary time dependence.

Without loss of generality, the function $\Phi(n)$ is submittable as follows

$$\Phi(n) = TU\exp[+\sigma n + \tau\,\mathrm{sh}\,\sigma] + TU^{-1}\exp[-\sigma n - \tau\,\mathrm{sh}\,\sigma]. \tag{70}$$

Here U is an arbitrary regular constant $M \times M$ matrix, while the c-number constant σ and the time-dependent $M \times M$ matrix T have to be specified substituting the ansatz (70) into equations (67) and (69). In particular, we obtain

$$2\,\mathrm{ch}\,\sigma = \gamma \tag{71}$$

$$\dot{T} = CT - T\,\mathrm{ch}\,\sigma. \tag{72}$$

Fortunately, the information encoded in equation (72) turns out to be quite unnecessary for practical purposes, irrespective of whether C or T was preferred to be pre-assigned. Indeed, according to the ansatz (65) and (66) and the formula (70), the required solution of the general model (12) can be presented in either of two equivalent forms

$$F(n) = \Psi^{-1}(n)\Psi(n+1)F \tag{73}$$

or

$$G(n) = G\Psi^{-1}(n+1)\Psi(n) \tag{74}$$

where

$$\Psi(n) = U\exp[+\sigma n + \tau\,\mathrm{sh}\,\sigma] + U^{-1}\exp[-\sigma n - \tau\,\mathrm{sh}\,\sigma] \tag{75}$$

and the matrix T is seen to be cancelled without trace.

We illustrate the result just generated (73)–(75) in terms of a four-component model ($M = 2$) parametrizing the matrices U and F by

$$U_{11} = \mathrm{ch}\,\eta\exp(+\delta + \theta) \tag{76}$$

$$U_{12} = \mathrm{sh}\,\eta\exp(+\delta + \varphi) \tag{77}$$

$$U_{21} = \mathrm{sh}\,\eta\exp(+\delta - \varphi) \tag{78}$$

$$U_{22} = \mathrm{ch}\,\eta\exp(+\delta - \theta) \tag{79}$$

and

$$F_{11} = \mathrm{ch}\,\rho\exp(+w + \alpha) \tag{80}$$

$$F_{12} = \mathrm{sh}\,\rho\exp(+w + \beta) \tag{81}$$

$$F_{21} = \mathrm{sh}\,\rho\exp(+w - \beta) \tag{82}$$

$$F_{22} = \mathrm{ch}\,\rho\exp(+w - \alpha) \tag{83}$$

respectively.

Then the matrix elements $F_{jk}(n)$ of the 2×2 matrix $F(n)$ can be expressed as

$$F_{jk}(n) = \frac{f_{jk}(n)\exp(+w)}{\mathrm{sh}^2 d(n) + \mathrm{ch}^2\eta\,\mathrm{ch}^2\theta} \tag{84}$$

where

$$d(n) = \sigma n + \tau\,\mathrm{sh}\,\sigma + \delta \tag{85}$$

while

$$f_{11}(n) = [\operatorname{sh} d(n) \operatorname{sh} d(n+1) + \operatorname{ch}^2\eta \operatorname{ch}\theta \operatorname{ch}(\sigma+\theta)] \operatorname{ch}\rho \exp(+\alpha) + \operatorname{ch}\eta \operatorname{sh}\eta \operatorname{ch}\theta \operatorname{sh}\sigma \operatorname{sh}\rho \exp(+\varphi-\beta) \tag{86}$$

$$f_{12}(n) = [\operatorname{sh} d(n) \operatorname{sh} d(n+1) + \operatorname{ch}^2\eta \operatorname{ch}\theta \operatorname{ch}(\sigma+\theta)] \operatorname{sh}\rho \exp(+\beta) + \operatorname{ch}\eta \operatorname{sh}\eta \operatorname{ch}\theta \operatorname{sh}\sigma \operatorname{ch}\rho \exp(+\varphi-\alpha) \tag{87}$$

$$f_{21}(n) = [\operatorname{sh} d(n) \operatorname{sh} d(n+1) + \operatorname{ch}^2\eta \operatorname{ch}\theta \operatorname{ch}(\sigma-\theta)] \operatorname{sh}\rho \exp(-\beta) + \operatorname{ch}\eta \operatorname{sh}\eta \operatorname{ch}\theta \operatorname{sh}\sigma \operatorname{ch}\rho \exp(-\varphi+\alpha) \tag{88}$$

$$f_{22}(n) = [\operatorname{sh} d(n) \operatorname{sh} d(n+1) + \operatorname{ch}^2\eta \operatorname{ch}\theta \operatorname{ch}(\sigma-\theta)] \operatorname{ch}\rho \exp(-\alpha) + \operatorname{ch}\eta \operatorname{sh}\eta \operatorname{ch}\theta \operatorname{sh}\sigma \operatorname{sh}\rho \exp(-\varphi+\beta). \tag{89}$$

To recover the final results for the field components $w(n)$, $\rho(n)$ and $\alpha(n)$, $\beta(n)$, we must simply substitute the obtained expressions (84)–(89) into the relations

$$w(n) = \frac{1}{2} \ln [F_{11}(n)F_{22}(n) - F_{12}(n)F_{21}(n)] \tag{90}$$

$$\rho(n) = \frac{1}{2} \ln \frac{\sqrt{f_{11}(n)f_{22}(n)} + \sqrt{f_{12}(n)f_{21}(n)}}{\sqrt{f_{11}(n)f_{22}(n)} - \sqrt{f_{12}(n)f_{21}(n)}} \tag{91}$$

$$\alpha(n) = \frac{1}{2} \ln \frac{f_{11}(n)}{f_{22}(n)} \tag{92}$$

$$\beta(n) = \frac{1}{2} \ln \frac{f_{12}(n)}{f_{21}(n)} \tag{93}$$

rendering the mere inversion of the earlier adopted parametrization (13)–(16) for the matrix elements $F_{jk}(n)$.

In general, the solutions found for the modes $\rho(n)$ and $\alpha(n)$, $\beta(n)$ scarcely permit further simplifications. However, the solution for the mode $w(n)$ can be readily arranged to yield

$$w(n) = \frac{1}{2} \ln \frac{\operatorname{sh}^2 d(n+1) + \operatorname{ch}^2\eta \operatorname{ch}^2\theta}{\operatorname{sh}^2 d(n) + \operatorname{ch}^2\eta \operatorname{ch}^2\theta} + w. \tag{94}$$

This expression resembles the solution

$$q(n) = \ln \frac{\operatorname{ch}[\sigma(n+1) + \tau \operatorname{sh}\sigma + \delta]}{\operatorname{ch}[\sigma n + \tau \operatorname{sh}\sigma + \delta]} + q \tag{95}$$

supported by the standard Toda lattice [4, 5].

Meanwhile, in the particular case of $\eta = 0$ and $\theta = 0$ the simplification turns out to be so drastic that the modes $\rho(n)$ and $\alpha(n)$, $\beta(n)$ stay totally frozen on their trivial values $\rho(n) = \rho$ and $\alpha(n) = \alpha$, $\beta(n) = \beta$ whereas the excited mode $w(n)$ plays the same role as a displacement mode $q(n)$ in the Toda model (1).

It is interesting to note another particular case $\rho = 0$ when only one mode $\beta(n)$ remains frozen, $\beta(n) = \varphi - \alpha$.

In a general case $\eta \neq 0$, $\theta \neq 0$, $\rho \neq 0$, when all four field components $w(n)$, $\rho(n)$, $\alpha(n)$, $\beta(n)$ are nontrivially excited, a partial analysis of obtained solutions can still be carried out. We will distinguish four types of real solutions due to four different assumptions about parameters σ and δ in terms of new real parameters $\mu > 0$ and $x(0)$.

Namely, at $\sigma = \mu$ and $\delta = -\mu/2 - \mu x(0)$ we have

$$\operatorname{sh} d(n) = \operatorname{sh}\left[\mu\left(n - \tfrac{1}{2}\right) + \tau \operatorname{sh}\mu - \mu x(0)\right] \tag{96}$$

that corresponds to the regular four-component solution although propagating with velocity $v = -(\operatorname{sh}\mu)/\mu$, i.e. in the negative direction of the coordinate axis. The component $w(n)$ can be identified with a kink since its limiting values at opposite infinities are different

$$\lim_{n\to-\infty} w(n) = w - \mu \tag{97}$$

$$\lim_{n\to+\infty} w(n) = w + \mu. \tag{98}$$

Each of the other three components $\rho(n)$, $\alpha(n)$, $\beta(n)$ can be treated as a soliton on a finite background [10, 11] inasmuch as the respective limiting values at both infinities coincide having the nonzero values

$$\lim_{n\to-\infty} \rho(n) = \lim_{n\to+\infty} \rho(n) = \rho \tag{99}$$

$$\lim_{n\to-\infty} \alpha(n) = \lim_{n\to+\infty} \alpha(n) = \alpha \tag{100}$$

$$\lim_{n\to-\infty} \beta(n) = \lim_{n\to+\infty} \beta(n) = \beta. \tag{101}$$

Further at $\sigma = \mu + \pi\mathrm{i}$ and $\delta = -\mu/2 - \mu x(0)$ we have

$$\operatorname{sh} d(n) = \cos(\pi n)\operatorname{sh}\left[\mu\left(n - \tfrac{1}{2}\right) - \tau \operatorname{sh}\mu - \mu x(0)\right] \tag{102}$$

that corresponds once again to the regular four-component solution although propagating with the opposite velocity $v = (\operatorname{sh}\mu)/\mu$, i.e. in the positive direction of the coordinate axis. The conclusions on the character of the field components $w(n)$, $\rho(n)$, $\alpha(n)$, $\beta(n)$ are proved to be the same as in the previous paragraph and we will not repeat them.

At $\sigma = \mu$, $\delta = -\mu/2 - \mu x(0) + \pi\mathrm{i}/2$ when

$$\operatorname{sh} d(n) = \mathrm{i}\operatorname{ch}\left[\mu\left(n - \tfrac{1}{2}\right) + \tau \operatorname{sh}\mu - \mu x(0)\right] \tag{103}$$

and at $\sigma = \mu + \pi\mathrm{i}$, $\delta = -\mu/2 - \mu x(0) + \pi\mathrm{i}/2$ when

$$\operatorname{sh} d(n) = \mathrm{i}\cos(\pi n)\operatorname{ch}\left[\mu\left(n - \tfrac{1}{2}\right) + \tau \operatorname{sh}\mu - \mu x(0)\right] \tag{104}$$

we come to the divergent (nonphysical) solutions that are important as an intermediate material in generating more complex regular solutions.

6. Conclusion

In summary, we have developed a class of multi-component nonlinear lattice systems that are the natural matrix extension of the Toda system. We have found the general matrix form of the Lax representation applicable to any particular model in a class irrespective of the number of its field components. Then, being convinced of the system's integrability we have managed to find the Bäcklund auto-transformation relating to the whole class under study. In passing we have developed the auxiliary integrable nonlinear model leading in a straightforward manner to the general matrix Bäcklund auto-transformation of interest. Relying upon the obtained Bäcklund transformation we have generated the simplest nontrivial solutions of a general matrix model and analyse them in the case of the four-component system. We have revealed that one of the field components in regular solutions is of kink origin while the other three are of the soliton-on-constant-background-type. The irregular solutions, despite their unphysical divergence, can be invoked in generating more complex regular solutions. Throughout this paper we have used the four-component system mainly for illustrative purposes although some particular results relating to the four-component version appear to have an independent interest. Thus, we have shown that the four-component model is a typical dynamical system permitting both Lagrangian and Hamiltonian formulations. The Lagrangian and Hamiltonian functions related to the four-component model have been explicitly presented. We have also proved that the Bäcklund transformation of the four-component system is a canonical one. The generating function of the corresponding canonical transformation was found.

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